1. Derive the adding formula for combining the reflection $R$ and transmission $T$ matrices from the interaction principles for two atmospheric layers.

We write down the interaction principles for the two layers

\[
I_0^+ = T_1^+ I_1^+ + R_1^+ I_0^- + S_1^+ \\
I_1^- = T_1^- I_0^- + R_1^- I_1^+ + S_1^-
\]

\[
I_1^+ = T_2^+ I_2^+ + R_2^+ I_1^- + S_2^+ \\
I_2^- = T_2^- I_1^- + R_2^- I_2^+ + S_2^-
\]

where layer 1 is above layer 2 and the incident radiance vectors on the combined layers are $I_0^-$ downwelling on the top of layer 1 and $I_2^+$ upwelling on the bottom of layer 2. To get the adding formula the radiance vectors in the interface between layers 1 and 2 are eliminated.

Since we are not interested in the adding formula for the source vectors we can set them to zero, which does not affect the combined reflection and transmission. Substituting in for $I_1^+$ and $I_1^-$ gives

\[
I_1^- = T_1^- I_0^- + R_1^- (T_2^+ I_2^+ + R_2^+ I_1^-) \\
I_1^+ = T_2^+ I_2^+ + R_2^+ (T_1^- I_0^- + R_1^- I_1^+)
\]

Solving for $I_1^-$ and $I_1^+$ gives

\[
I_1^- = \Gamma^- [T_1^- I_0^- + R_1^- T_2^+ I_2^+] \\
I_1^+ = \Gamma^+ [T_2^+ I_2^+ + R_2^+ T_1^- I_0^-] \\
\Gamma^- = [1 - R_1^- R_2^+]^{-1} \\
\Gamma^+ = [1 - R_2^+ R_1^-]^{-1}
\]

Substituting in for $I_1^-$ and $I_1^+$ in the upwelling layer 1 and the downwelling layer 2 interaction principle equations gives

\[
I_0^+ = T_1^+ \Gamma^+ [T_2^+ I_2^+ + R_2^+ T_1^- I_0^-] + R_1^+ I_0^- \\
I_2^- = T_2^- \Gamma^- [T_1^- I_0^- + R_1^- T_2^+ I_2^+] + R_2^- I_2^+
\]

which can be rearranged to

\[
I_0^+ = [T_1^+ \Gamma^+ T_2^+] I_2^+ + [R_1^+ + T_1^+ \Gamma^+ R_2^+ T_1^-] I_0^- \\
I_2^- = [T_2^- \Gamma^- T_1^-] I_0^- + [R_2^- + T_2^- \Gamma^- R_1^- T_2^+] I_2^+
\]

Therefore, the combined reflection and transmission matrices are

\[
R_T^+ = R_1^+ + T_1^+ \Gamma^+ R_2^+ T_1^- \\
T_T^+ = T_1^+ \Gamma^+ T_2^+ \\
R_T^- = R_2^- + T_2^- \Gamma^- R_1^- T_2^+ \\
T_T^- = T_2^- \Gamma^- T_1^- 
\]